

Solutions to Maths workbook – 1 | Trigonometric Identities and Equations

Level - 3

Daily Tutorial Sheet - 16

174. We have, $\tan^4 x + \tan^4 y + 2 \cot^2 x \cot^2 y = 3 + \sin^2(x+y)$

$$\Rightarrow \tan^4 x + \tan^4 y + 2 \cot^2 x \cot^2 y - 2 = 1 + \sin^2(x+y)$$

$$\Rightarrow (\tan^2 x - \tan^2 y)^2 + 2(\tan x \tan y - \cot x \cot y)^2 = -1 + \sin^2(x+y)$$

Clearly, L.H.S ≥ 0 and R.H.S ≤ 0

$$\therefore \text{L.H.S} = \text{R.H.S} = 0$$

$$\Rightarrow \tan^2 x = \tan^2 y, \tan x \tan y = \cot x \cot y \text{ and } \sin^2(x+y) = 1$$

$$\Rightarrow \tan^2 x = \tan^2 y, \tan^2 x \tan^2 y = 1 \text{ and } \sin^2(x+y) = 1$$

$$\Rightarrow \tan^2 x = 1 \text{ and } x+y = n\pi \pm \frac{\pi}{2}, n, m, \in \mathbb{Z} \Rightarrow x = m\pi \pm \frac{\pi}{4} \text{ and } y = (n-m)\pi \mp \frac{\pi}{4}, n, m, \in \mathbb{Z}$$

$$\Rightarrow \text{Thus, } x = y = p\pi \pm \frac{\pi}{4}, p \in \mathbb{Z}$$

175. Here $\left[2 \sin \left(x + \frac{\pi}{6} \right) \right]^{\sqrt{2 \sin^2 x + 2\sqrt{3} \sin x \cdot \cos x + 1}} = 4 \Rightarrow \left[2 \sin \left(x + \frac{\pi}{6} \right) \right]^{\sqrt{3 \sin^2 x + \cos^2 x + 2\sqrt{3} \sin x \cdot \cos x}} = 4$

$$\Rightarrow \left[2 \sin \left(x + \frac{\pi}{6} \right) \right]^{\sqrt{3 \sin x + \cos x}} = 4$$

To make the R.H.S. well defined it is necessary that, $\sin \left(x + \frac{\pi}{6} \right) > 0 \Rightarrow x \in \left(-\frac{\pi}{6}, \frac{5\pi}{6} \right)$

Now, we have $\left\{ \sin \left(x + \frac{\pi}{6} \right) \right\}^{\sin(x+\pi/6)} = 1 \Rightarrow \sin \left(x + \frac{\pi}{6} \right) = 1 \Rightarrow x = \frac{\pi}{3}$

176. The given equation can be rewritten as $2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right) + 2 \cos^2 \left(\frac{x+y}{2} \right) - 1 = -\frac{3}{2}$

$$\Rightarrow 4 \cos^2 \left(\frac{x+y}{2} \right) + 4 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right) + 1 = 0 \Rightarrow \left(2 \cos \left(\frac{x+y}{2} \right) + \cos \left(\frac{x-y}{2} \right) \right)^2 + \sin^2 \left(\frac{x-y}{2} \right) = 0$$

$$\Rightarrow 2 \cos \left(\frac{x+y}{2} \right) = -\cos \left(\frac{x-y}{2} \right) \text{ and } \sin^2 \left(\frac{x-y}{2} \right) = 0$$

From second equation, we get $x-y = 2n\pi$, where $n \in I$ or $y = x - 2n\pi$

substituting the values of y in the first equation, we get $2 \cos(x - n\pi) = -\cos n\pi$

$$\Rightarrow 2 \cos x \cdot \cos n\pi = -\cos n\pi \Rightarrow \cos x = -\frac{1}{2} = \cos \left(\pi - \frac{\pi}{3} \right) \Rightarrow x = 2m\pi \pm \frac{2\pi}{3}, m \in I$$

$$\therefore \text{Solutions are } \left\{ \left(2m\pi \pm \frac{2\pi}{3} \right), \left(2(m-n)\pi \pm \frac{2\pi}{3} \right) \right\}, m, n \in I$$

177. We have, $\sec \theta + \operatorname{cosec} \theta = c \Rightarrow \sin \theta + \cos \theta = c \sin \theta \cdot \cos \theta$

Squaring both sides, we get $1 + 2 \sin \theta \cos \theta = c^2 \sin^2 \theta \cdot \cos^2 \theta \Rightarrow 1 + \sin 2\theta = \frac{c^2}{4} \sin^2 2\theta$

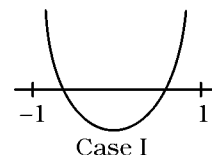
Put $\sin 2\theta = t \Rightarrow c^2 t^2 - 4t - 4 = 0$

Thus, the equation must have atleast one root in the interval $[-1, 1]$

Case I: $D \geq 0, f(-1) > 0, f(1) > 0$

$$\Rightarrow 16 + 16c^2 \geq 0, c^2 + 4 - 4 > 0, c^2 - 4 - 4 > 0 \Rightarrow c^2 > 8$$

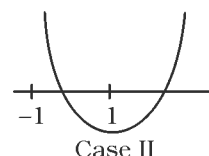
In this case we will get two distinct values of $\sin 2\theta$, resulting in 4 distinct values of $\theta \in (0, 2\pi)$ two of them would be repeated.



Case I

Case II: $f(1)f(-1) < 0 \Rightarrow (c^2 - 8)c^2 < 0 \Rightarrow c^2 < 8$

In this case we will get one value of $\sin 2\theta$, resulting in 2 distinct values of $\theta \in (0, 2\pi)$.



Case II

178. The given equation can be rewritten as $(\sin^4 x + \cos^4 x)^2 - 2\sin^4 x \cos^4 x = \frac{17}{16}(1 - \sin^2 2x)$

Put $t = \sin x \cos x, (1 - 2t^2)^2 - 2t^4 = \frac{17}{16}(1 - 4t^2) \Rightarrow 32t^4 + 4t^2 - 1 = 0 \Rightarrow 32t^4 + 8t^2 - 4t^2 - 1 = 0$

$$\Rightarrow (4t^2 + 1)(8t^2 - 1) = 0 \Rightarrow t = \pm \frac{1}{2\sqrt{2}} \quad [\because 4t^2 + 1 \neq 0]$$

$$\Rightarrow \sin x \cos x = \pm \frac{1}{2\sqrt{2}} \Rightarrow \sin 2x = \pm \frac{1}{\sqrt{2}} \Rightarrow 2x = n\pi \pm (-1)^n \left(\frac{\pi}{4}\right) \Rightarrow x = \frac{n\pi}{2} \pm (-1)^n \frac{\pi}{8}$$

$$\Rightarrow x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8} \text{ and } \frac{7\pi}{8}$$

179. $\cos^2 x = \frac{(a+2) \pm \sqrt{(a+2)^2 + 4(a+3)}}{2} \Rightarrow \cos^2 x = \frac{(a+2) \pm (a+4)}{2}$

Either, $\cos^2 x = -1$ (not possible) or $\cos^2 x = a + 3$

since, $0 \leq \cos^2 x \leq 1 \Rightarrow 0 \leq a + 3 \leq 1 \Rightarrow -3 \leq a \leq -2$

Also, $\cos^2 x = (a + 3) \Rightarrow x = n\pi \pm \cos^{-1} \sqrt{a+3}$, where $n \in I$ and $a \in [-3, -2]$